SPRING 2025 MATH 540: QUIZ 5

Name:

1. Let X be a set and $\sim$ a relation on X. Define what it means for $\sim$ to be an equivalence relation. (3 points)

Solution. The relation $\sim$ is an equivalence relation on X is it satisfies:

(i) $x \sim x$, for all $x \in X$.
(ii) If $x \sim y$, then $y \sim x$, for all $x, y \in X$.
(iii) If $x \sim y$ and $y \sim z$, then $x \sim z$, for all $x, y, z \in X$.

2. For the set $\{(a,b) \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$ discussed in class, with equivalence classes denoted $[(a,b)]$ show that multiplication of equivalence classes given by $[(a,b)] \cdot [(c,d)] = [(ad+bc, bd)]$ is well defined. (4 points)

Solution. Suppose $[a,b)] = [(a',b')]$ and $[(c,d)] = [(c',d,)]$. Then, $ab' - a'b = 0$ and $cd' - c'd = 0$. Multiplying the first equation by cd' and the second equation by $a'b$ and adding we get $acb'd' - a'c'bd = 0$, so that $[(ac, bd)] = [(a'b', c'd')]$, which is what we want.

3. Find all solutions to the linear congruences $6x \equiv 21 \bmod 27$, both in $\mathbb{Z}_{27}$ and in $\mathbb{Z}$.

Solution. We have $\gcd(6, 27) = 3$ which divides 21, so we expect 3 solutions in $\mathbb{Z}_{27}$. Dividing the original congruence by 3 we have $2x \equiv 7 \bmod 9$, which has 8, as an element of $\mathbb{Z}_{27}$ as a solution. The other solutions are $8 + \frac{27}{3} = 17$ and $8 + 2 \cdot \frac{27}{3} = 26$ as elements of $\mathbb{Z}_{27}$. As elements of $\mathbb{Z}$ the solutions are

$$\{27n + 8 \mid n \in \mathbb{Z}\} \cup \{27n + 17 \mid n \in \mathbb{Z}\} \cup \{27n + 26 \mid n \in \mathbb{Z}\} = \{27n + 8 \mid n \in \mathbb{Z}\}$$